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# On some subspaces of an FK-space

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**Abstract**. *In this paper we study the subspaces $C_1S$, $C_1W$, $C_1F$ and $C_1B$ for a locally convex FK-space $X$ containing $\phi$, the space of finite sequences.*

**Key words:** *FK-space, AK-space, $\sigma K$-space, $\sigma B$-space, $C_1-$ summability method*

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## 1. Introduction and notation

Let $w$ denote the space of all complex-valued sequences. An FK-space is a locally convex vector subspace of $w$ which is also a Fréchet space (complete linear metric) with continuous coordinates. A BK-space is a normed FK-space. The basic properties of FK-spaces may be found in [7], [8] and [10]. We now define the Cesáro summability matrix which is used throughout this paper: The Cesáro mean is given by the matrix $C_1$ whose $nkth$ entry is

$$C_1[n,k] = \begin{cases} \frac{1}{n+1}, \text{ if } 0 \leq k \leq n \\ 0, \text{ if } k > n. \end{cases}$$

The sequence spaces

$$\sigma_0 = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{j=1}^{n} x_j = 0 \right\},$$

$$\sigma b = \left\{ x \in w : \sup_n \left| \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j \right| < \infty \right\}$$

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and

$$\sigma s=\left\{x\in w:\lim_{n}\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_j\text{ exists}\right\}$$

are BK-spaces with the norm

$$\|x\|_{\sigma_0}=\sup_n\frac{1}{n}\left|\sum_{k=1}^{n}x_k\right|$$

and

$$\|x\|_{\sigma s}=\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_j\right|$$

respectively ([1], [2] and [9]).

Throughout the paper $\delta^j$, $(j=1,2,...)$, the sequence $(0,0,...,0,1,0,...)$ with the one in the $j$-th position; $\phi$ the linear span of the $\delta^j$ 's. The topological dual of $X$ is denoted by $X'$. A sequence $x$ in a locally convex sequence space $X$ is said to have the property AK (respectively $\sigma K$) if $x^{(n)}\to x$ (respectively $\frac{1}{n}\sum\limits_{k=1}^{n}x^{(k)}\to x$) in $X$ where $x^{(n)}=(x_1,x_2,...,x_n,0,...)=\sum\limits_{k=1}^{n}x_k\delta^k$. It is known that if an FK-space $\phi\subset X$ is said to have $\sigma B$ if $\left\{\frac{1}{n}\sum\limits_{k=1}^{n}x^{(k)}\right\}$ is a bounded set in $X$ for each $x\in X$. Also, an FK- space $X$ is said to have $F\sigma K$ ( functional $\sigma K$ ) if $X\subset C_1F^{+}$ i.e., $X=C_1F$ ([1], [2] and [4]).

We recall (see [3] and [4] ) that the $f,\sigma-$ and $\sigma b-$ duals of a subset $X$ of $w$ are defined to be

$$X^f=\left\{\left\{f\left(\delta^k\right)\right\}:f\in X'\right\},$$

$$\begin{aligned}X^{\sigma}&=\left\{x\in w:\lim_{n}\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_jy_j\text{ exists for all }y\in X\right\}\\&=\left\{x\in w:x.y\in\sigma s\text{ for all }y\in X\right\},\end{aligned}$$

$$\begin{aligned}X^{\sigma b}&=\left\{x\in w:\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_j\right|<\infty\text{ for all }y\in X\right\}\\&=\left\{x\in w:x.y\in\sigma b\text{ for all }y\in X\right\},\end{aligned}$$

respectively, where $x.y=(x_ny_n)$.

## 2. Some subspaces of $X$

Following [4] we recall some important subspaces of a locally convex FK-space $X$ containing $\phi$.

**Definition1.** *Let $X$ be an FK-space$\supset \phi$. Then*

$$W := W(X) = \left\{ x \in X : x^{(k)} \to x(\textit{weakly}) \textit{ in } X \right\}$$

$$\begin{aligned}
C_1W := \quad C_1W(X) \quad &= \left\{ x \in X : \tfrac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x(\textit{weakly}) \textit{ in } X \right\} \\
&= \left\{ x \in X : f(x) = \lim_n \tfrac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j f\left(\delta^j\right) \textit{ for all } f \in X' \right\} \\
&= \left\{ x \in X : x \textit{ has } S\sigma K \textit{ in } X \right\}, \\
C_1S := \quad C_1S(X) \quad &= \left\{ x \in X : \tfrac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x \right\} \\
&= \left\{ x \in X : x \textit{ has } \sigma K \textit{ in } X \right\} \\
&= \left\{ x \in X : x = \lim_n \tfrac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j \delta^j \right\}, \\
C_1F^+ := \quad C_1F^+(X) \quad &= \left\{ x \in w : \left( \tfrac{1}{n} \sum_{k=1}^{n} x^{(k)} \right) \text{ is } \textit{weakly Cauchy in } X \right\} \\
&= \left\{ x \in w : \left(x_n f\left(\delta^n\right)\right) \in \sigma s \textit{ for all } f \in X' \right\}, \\
C_1B^+ := \quad C_1B^+(X) \quad &= \left\{ x \in w : \left( \tfrac{1}{n} \sum_{k=1}^{n} x^{(k)} \right) \text{ is } \textit{bounded in } X \right\} \\
&= \left\{ x \in w : \left(x_n f\left(\delta^n\right)\right) \in \sigma b \textit{ for all } f \in X' \right\},
\end{aligned}$$

*also*

$$C_1F := \quad C_1F^+ \cap X \qquad \textit{and} \quad C_1B := C_1B^+ \cap X.$$

We note that subspaces $W$ and $C_1W$ are closely related to conullity and Cesáro conullity of the $FK-$space $X$ ( see [5] and [6] ).

We now study some inclusions which are analogous to those given in [8; *Chapter* 10] .

**Theorem 2.** *Let $X$ be an FK-space$\supset \phi$. Then*

$$\phi \subset C_1S \subset C_1W \subset C_1F \subset C_1B \subset X \textit{ and } \phi \subset C_1S \subset C_1W \subset \overline{\phi}.$$

**Proof.** The only non-trivial part is $C_1W \subset \overline{\phi}$. Let $f \in X'$ and $f = 0$ on $\phi$. The definition of $C_1W$ shows that $f = 0$ on $C_1W$. Hence, the Hahn-Banach theorem gives the result. □

**Theorem 3.** *The subspaces $E = C_1S$, $C_1W$, $C_1F$, $C_1F^+$, $C_1B$, and $C_1B^+$ of $X$ FK-space are monotone i.e., if $X \subset Y$ then $E(X) \subset E(Y)$.*

**Proof.** The inclusion map $i : X \to Y$ is continuous by Corollary 4.2.4 of [8], so $\frac{1}{n} \sum_{k=1}^{n} x^{(k)} \to x$ in $X$ implies the same in $Y$. This proves the assertion for $C_1S$. For $C_1W$ it follows from the fact that $i$ is weakly continuous by (4.0.11) of [8]. Now

$z \in C_1F^+, C_1B^+$ if and only if $(z_n f(\delta^n)) \in \sigma s, \sigma b$ respectively for all $f \in X'$, hence for all $g \in Y'$ since $g|X \in X'$ by Corollary 4.2.4 of [8]. The result follows for $C_1F^+$, $C_1B^+$ and so for $C_1F$, $C_1B$. □

Since $\sigma_0$ is an $AK-$space, we immediately get the following

**Theorem 4.** *Let $X$ be an FK-space$\supset \sigma_0$. Then $\sigma_0 \subset C_1S \subset C_1W$.*

**Theorem 5.** *Let $X$ be an FK-space$\supset \phi$. Then $C_1B^+ = X^{f\sigma b}$.*

**Proof.** By *Definition 1*, $z \in C_1B^+$ if and only if $z.u \in \sigma b$ for each $u \in X^f$. This is precisely the assertion. □

**Theorem 6.** *Let $X$ be an FK-space$\supset \phi$. Then $C_1B^+$ is the same for all FK-spaces $Y$ between $\overline{\phi}$ and $X$; i.e., $\overline{\phi} \subset Y \subset X$ implies $C_1B^+(Y) = C_1B^+(X)$. Here the closure of $\phi$ is calculated in $X$.*

**Proof.** By *Theorem 3* we have $C_1B^+(\overline{\phi}) \subset C_1B^+(Y) \subset C_1B^+(X)$. By *Theorem 5* and by (7.2.4) of [8] the first and the last are equal. □

**Theorem 7.** *Let $X$ be an FK-space such that $C_1B \supset \overline{\phi}$. Then $\overline{\phi}$ has $\sigma K$ and $C_1S = C_1W = \overline{\phi}$.*

**Proof.** Suppose first that $X$ has $\sigma B$. Define $f_n : X \to X$ by

$$f_n(x) = x - \frac{1}{n}\sum_{k=1}^{n} x^{(k)}.$$

Then $\{f_n\}$ is pointwise bounded, hence equicontinuous by (7.0.2) of [8]. Since $f_n \to 0$ on $\phi$ then also $f_n \to 0$ on $\overline{\phi}$ by (7.0.3) of [8]. This is the desired conclusion. □

**Theorem 8.** *Let $X$ be an FK-space$\supset \phi$. Then $C_1F^+ = X^{f\sigma}$.*

**Proof.** This may be proved as in *Theorem 5*, with $\sigma s$ instead of $\sigma b$. □

**Theorem 9.** *Let $X$ be an FK-space$\supset \phi$. Then $C_1F^+$ is the same for all FK-spaces $Y$ between $\overline{\phi}$ and $X$; i.e., $\overline{\phi} \subset Y \subset X$ implies $C_1F^+(Y) = C_1F^+(X)$.(The closure of $\phi$ is calculated in $X$).*
The proof is similar to that of *Theorem 6*.

**Lemma 10.** *Let $X$ be an FK-space in which $\overline{\phi}$* has $\sigma K$. *Then $C_1F^+ = (\overline{\phi})^{\sigma\sigma}$.*

**Proof.** Observe that $C_1F^+ = X^{f\sigma}$ by *Theorem 8*. Since $X^f = (\overline{\phi})^f$ by Theorem 7.2.4 of [8], we have $X^{f\sigma} = (\overline{\phi})^{f\sigma}$. Hence, by Theorem 1.9 of [4] the result follows.□

**Theorem 11.** *Let $X$ be an FK-space$\supset \phi$. Then $X$ has $F\sigma K$ if and only if $\overline{\phi}$ has $\sigma K$ and $X \subset (\overline{\phi})^{\sigma\sigma}$.*

**Proof.** *Necessity.* $X$ has $\sigma B$ since $C_1F \subset C_1B$ so $\overline{\phi}$ has $\sigma K$ by *Theorem 7*. The remainder of the proof follows from *Lemma 10*. Sufficiency is given by *Lemma 10*.□

**Theorem 12.** *Let $X$ be an FK-space$\supset \phi$. The following are equivalent:*
*(i) $X$ has $F\sigma K$,*

$(ii)$ $X \subset C_1 S^{\sigma\sigma}$,
$(iii)$ $X \subset C_1 W^{\sigma\sigma}$,
$(iv)$ $X \subset C_1 F^{\sigma\sigma}$,
$(v)$ $X^{\sigma} = C_1 S^{\sigma}$,
$(vi)$ $X^{\sigma} = C_1 F^{\sigma}$.

**Proof.** Observe that $(ii)$ implies $(iii)$ and $(iii)$ implies $(iv)$ and that they are trivial since

$$C_1 S \subset C_1 W \subset C_1 F.$$

If $(iv)$ is true, then $X^{f} \subset C_1 F^{\sigma} = X^{f\sigma\sigma} \subset X^{\sigma}$ so $(i)$ is true by Theorem 1.9 of [4]. If $(i)$ holds, then *Theorem 11* implies that $\overline{\phi} = C_1 S$ and that $(ii)$ holds. The equivalence of $(v), (vi)$ with the others is clear. □

**Theorem 13.** *Let $X$ be an FK-space $\supset \phi$. The following are equivalent:*
$(i)$ *$X$ has $S\sigma K$,*
$(ii)$ *$X$ has $\sigma K$,*
$(iii)$ $X^{\sigma} = X'$.

**Proof.** Clearly $(ii)$ implies $(i)$. Conversely if $X$ has $S\sigma K$ it must have AD for $C_1 W \subset \overline{\phi}$ by *Theorem 2*. It also has $\sigma B$ since $C_1 W \subset C_1 B$. Thus $X$ has $\sigma K$ by *Theorem 7*, this proves that $(i)$ and $(ii)$ are equivalent. Assume that $(iii)$ holds. Let $f \in X'$, then there exists $u \in X^{\sigma}$ such that

$$f(x) = \lim_{n} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} u_j x_j$$

for $x \in X$. Since $f\left(\delta^{j}\right) = u_j$, it follows that each $x \in C_1 W$ which shows that $(iii)$ implies $(i)$. That $(ii)$ implies $(iii)$ is known (see [2], page 97). □

**Theorem 14.** *Let $X$ be an FK-space $\supset \phi$. The following are equivalent:*
$(i)$ *$C_1 W$ is closed in $X$,*
$(ii)$ $\overline{\phi} \subset C_1 B$,
$(iii)$ $\overline{\phi} \subset C_1 F$,
$(iv)$ $\overline{\phi} = C_1 W$,
$(v)$ $\overline{\phi} = C_1 S$,
$(vi)$ *$C_1 S$ is closed in $X$.*

**Proof.** $(ii)$ implies $(v)$: By *Theorem 7*, $\overline{\phi}$ has $\sigma K$, *i.e.* $\overline{\phi} \subset C_1 S$. The opposite inclusion is *Theorem 2*. Note that $(v)$ implies $(iv)$, $(iv)$ implies $(iii)$ and $(iii)$ implies $(ii)$ because

$$C_1 S \subset C_1 W \subset \overline{\phi},\ C_1 W \subset C_1 F \subset C_1 B;$$

$(i)$ implies $(iv)$ and $(vi)$ implies $(v)$ since $\phi \subset C_1 S \subset C_1 W \subset \overline{\phi}$. Finally $(iv)$ implies $(i)$ and $(v)$ implies $(vi)$. □

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